

MATHEMATICAL ASPECTS OF FINANCIAL MARKETS WITH FRICTIONS

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Academic dissertation

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Cover photo: *Stock trading on the New York Curb Association market, with brokers and clients signalling from street to offices (1916)*. Source: *George Grantham Bain Collection (Library of Congress)*.

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I should mention here that my decision to focus on stochastics in my studies was, in fact, influenced by Tommi to some extent. In the spring of 2003, as a first-year student majoring in economics, I took as a part of the statistics curriculum an introductory probability course, lectured by him. Near the end of the course, he digressed from the core content and talked about some more—or perhaps *too*—advanced topics related to *martingales*, involving e.g. σ -algebras, *stopping times*, and *Doob's optional sampling theorem*. At that time, my meagre mathematical background consisted of a basic analysis course, so obviously I was not quite able to grasp the subtleties of the theory. Yet I was somehow fascinated by these concepts and I determined to understand them better. In some sense, this dissertation is an outcome of the project.

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Mikko Pakkanen

Overview of the dissertation

This dissertation concerns certain mathematical questions that pertain to models of financial markets that involve *frictions*, i.e. factors that hinder trading of securities. The work consists of an introductory part and of the following three research papers:

- [I] PAKKANEN, M. S. (2010). [Microfoundations for diffusion price processes](#). *Mathematics and Financial Economics* **3**, 89–114.
- [II] PAKKANEN, M. S. (2010). [Stochastic integrals and conditional full support](#). *Journal of Applied Probability* **47**, 650–667.
- [III] PAKKANEN, M. S. (2010). [Brownian semistationary processes and conditional full support](#). Submitted for publication.

The first chapter of the introductory part begins with an example that elucidates the relevance of frictions in financial modeling, and then briefly introduces the fields of research to which the papers above are related to. The second chapter is a summary of the key mathematical methods and concepts that are used in the papers. Finally, an appendix provides a proof of a lemma in [I], omitted from the original paper, and a slight generalization of a result in [II].

Notations and nomenclature

SETS OF NUMBERS AND MATRICES. We use the set theoretic convention $\mathbb{N} := \{0, 1, 2, \dots\} \supsetneq \mathbb{Z}_+ := \{1, 2, \dots\}$. We denote the space of $d \times d$ matrices over \mathbb{R} is by $M_d(\mathbb{R})$ and the transpose of $A \in M_d(\mathbb{R})$ by A^T .

MEASURE THEORY. When X is a topological space, $\mathcal{B}(X)$ stands for the Borel σ -algebra, generated by the open subsets of X . The shorthand ‘meas’ refers to the Lebesgue measure on \mathbb{R} .

PROBABILITY THEORY. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and (E, \mathcal{E}) a measurable space. When $X : \Omega \longrightarrow E$ is a measurable map, that is a *random element* in E , we denote by $\mathcal{L}[X]$ the push-forward measure $\mathbf{P} \circ X^{-1}$ on (E, \mathcal{E}) , which we call the *law* of X .

When \mathbf{Q} is another probability measure on (Ω, \mathcal{F}) , and $\mathbf{P}[A] > 0$ if and only if $\mathbf{Q}[A] > 0$, $A \in \mathcal{F}$, we say that \mathbf{P} and \mathbf{Q} are *equivalent* and write $\mathbf{P} \sim \mathbf{Q}$.

Notations $\text{U}(0, 1)$ and $\text{Exp}(1)$ stand, respectively, for the uniform distribution on $(0, 1)$ and for the exponential distribution with rate parameter 1.

Let $\mathbb{T} \subset \mathbb{R}$ be an interval. When $(X_t)_{t \in \mathbb{T}}$ is a continuous-time stochastic process on $(\Omega, \mathcal{F}, \mathbf{P})$, we denote by $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \in \mathbb{T}}$ its natural filtration that is augmented the usual way to make it right-continuous and completed with \mathbf{P} -null sets (see [20, p. 124]).

FUNCTION SPACES. Let $0 \leq u < v < \infty$ and let $I \subset \mathbb{R}$ be an interval. We denote by $C([u, v], I)$ the space of continuous functions $[u, v] \longrightarrow I$, equipped with the usual uniform topology induced by the sup norm $\|\cdot\|_\infty$. For any $x \in I$, we denote by $C_x([u, v], I)$ the space of functions $f \in C([u, v], I)$ such that $f(u) = x$. When $I = \mathbb{R}$, we simply write $C([u, v])$ and $C_x([u, v])$ instead of $C([u, v], \mathbb{R})$ and $C_x([u, v], \mathbb{R})$, respectively.

For the space of *càdlàg* functions (i.e. right-continuous functions with finite left-hand limits) from $[0, \infty)$ to $S \subset \mathbb{R}^d$, we use the conventional notation $D([0, \infty), S)$.

MISCELLANEOUS. We write $f(x) \sim g(x)$ if $f(x) = g(x)(1 + o(1))$, and we use the shorthand ‘u.o.c.’ for ‘uniformly on compact sets’.

Contents

1	Financial markets with and without frictions	1
1.1	Why should we worry about frictions?	1
1.2	Approximation of price dynamics in markets with frictions . .	3
1.3	Arbitrage and hedging with small transaction costs	6
2	Mathematical concepts and methods	15
2.1	Supports of the laws of stochastic processes	15
2.2	The conditional full support property	18
2.3	Pure-jump Markov processes and their approximation	20
2.4	Heavy tails and regular variation	23
A	Appendix	25
A.1	Proof of Lemma 2.1 of [I]	25
A.2	Extension of Theorem 3.3 of [II]	26
	Bibliography	27
	Included papers [I, II, III]	

1

Financial markets with and without frictions

1.1 Why should we worry about frictions?

Let us consider a motivating example from option pricing. Suppose that an investor has written a European call option with strike price $K \in (0, \infty)$ and maturity $T \in (0, \infty)$ on a stock, whose price per share follows a geometric Brownian motion. This entails that at the time T the trader is exposed to liability

$$(S_T - K)^+, \quad (1.1)$$

denoting by S_T the terminal value of the stochastic process

$$S_t := s_0 e^{W_t - t/2}, \quad t \in [0, T],$$

where $s_0 \in (0, \infty)$ is a constant and $(W_t)_{t \in [0, T]}$ is a standard Brownian motion. For simplicity, let us assume that the interest rate is zero.

Suppose that the writer has no taste for taking risks. She wants to *superhedge* her exposure to (1.1). Thus, prior to writing the call option, she has collected a premium $c \in (0, \infty)$ from the buyer of the option, so that for some predictable trading strategy $(\varphi_t)_{t \in [0, T]}$, where φ_t denotes the position in the underlying stock at time t , one has

$$(S_T - K)^+ \leq c + \int_0^T \varphi_t dS_t. \quad (1.2)$$

From the classical results of Black and Scholes [6] and Merton [22], we know that such a strategy exists. To be specific, if $c := \mathbf{E}[(S_T - K)^+]$, $\varphi_t := \psi(S_t, t)$ for all $t \in [0, T)$, and $\varphi_T := \varphi_{T-}$, with

$$\psi(x, y) := \Phi\left(\frac{\log(x/K) + (T - y)/2}{\sqrt{T - y}}\right), \quad (x, y) \in [0, \infty) \times [0, T),$$

where Φ stands for the cumulative distribution function of the standard Gaussian distribution, then (1.2) holds with an equality. Thus, the strategy φ , known as the *delta hedge*, instead of merely superhedging, *replicates* the call option perfectly.

Unfortunately, viability of the delta hedge φ is highly contingent upon the idealized assumptions of the model, which we have not spelled out yet. To begin with, we should appreciate the fact that, from an economic point of view, perfect replication would be too good to be true. Hakansson [15, pp. 722–724] argues that if perfect replication of an option was indeed possible, then the option would be a *redundant* asset and no (reasonably) rational trader should be willing to buy one. Hence the empirical fact that options are traded in real markets would not make much sense. This conundrum is explained by the assumption that trading takes place in a *frictionless* market where, in Merton’s [22, p. 162] words,

“...there are no transactions [sic] costs or differential taxes. Trading takes place continuously and borrowing and short-selling are allowed without restriction.”

Also Black and Scholes [6, p. 640] introduce similar assumptions in their formulation.

Let us elaborate on some of the properties of the strategy φ , which are unreasonable from the point of view of real-world trading that involves *frictions*. In reality, only *piecewise-constant* strategies, i.e. ones that require at most finitely many trades, are possible. The delta hedge φ clearly does not fit the bill. Even if we gloss over this issue, there are still problems. Since the function ψ is smooth on the domain $(0, \infty) \times (0, T)$, an application of Itô’s formula yields for any $t \in (0, T)$ and $\varepsilon \in (0, T - t)$,

$$\begin{aligned} \varphi_{t+\varepsilon} - \varphi_t &= \int_t^{t+\varepsilon} \frac{\partial \psi}{\partial y}(S_u, u) du + \int_t^{t+\varepsilon} \frac{\partial \psi}{\partial x}(S_u, u) dS_u \\ &\quad + \frac{1}{2} \int_t^{t+\varepsilon} \frac{\partial^2 \psi}{\partial x^2}(S_u, u) S_u^2 du. \end{aligned}$$

Hence, as $\partial \psi / \partial x$ is non-vanishing, we have

$$\langle \varphi, \varphi \rangle_{t+\varepsilon} - \langle \varphi, \varphi \rangle_t = \int_t^{t+\varepsilon} \left(\frac{\partial \psi}{\partial x}(S_u, u) S_u \right)^2 du > 0, \quad (1.3)$$

where $\langle \varphi, \varphi \rangle$ denotes the *quadratic variation* of φ . Thus, the strategy φ has positive quadratic variation on *any* subinterval of $[0, T]$. However, in reality the number of shares held at the moment equals the number of shares bought so far subtracted by the number of shares sold so far. This entails that

$$\varphi_t = \varphi_t^b - \varphi_t^s, \quad t \in [0, T],$$

where $(\varphi_t^b)_{t \in [0, T]}$ and $(\varphi_t^s)_{t \in [0, T]}$ are processes that count the shares bought and sold so far, respectively. By their very nature, the processes φ^b and φ^s are non-decreasing, which immediately implies that φ is of *finite variation*, which in turn is, of course, at odds with (1.3). This conflict can be

interpreted so that the delta hedge requires ability to trade infinitely many shares of the stock during any (arbitrarily short) period. Any transaction costs, or indeed the sheer physical impossibility of trading infinitely many shares, thus render the strategy infeasible.

Moreover, the delta hedge φ hinges upon the requirement that at any time $t \in [0, T]$, it is possible to trade any number of shares at the same *exogenous* unit price S_t . In reality, however, buying (resp. selling) a larger number of shares raises (resp. lowers) the unit price of the stock, i.e. supply and demand of the stock are not perfectly elastic. Depending on the liquidity of the stock, this may become an issue when one needs to delta hedge, as is usually the case in practice, a larger position in calls. Finally, sometimes it is not possible to trade at all—say, at night when the market is closed—while at the same time information, e.g. rumors of a take-over bid, that is bound to affect the price of the stock when trading resumes might surface. In such a situation, the price may shoot up between the occasions the writer is able to update the delta hedge, causing her to incur *a priori* unbounded losses.

Thus, we have seen that at least the following frictions prevent the trader from replicating a call option perfectly in real markets:

- impossibility of continuous trading,
- impossibility of trading infinitely many shares/transaction costs,
- illiquidity/inelastic supply and demand of the stock.

(Of course this is not an exhaustive list of possible frictions. Frictions do exist also on the money market side, and in the form of taxes.) The failure of the delta hedge has the imminent consequence that any investor who writes a call option must accept risk of incurring losses or raise the premium collected from the buyer (there is an obvious trade-off between these, non-exclusive, choices).

This example from option pricing suggests that frictions and their effects should be properly recognized when quantitative financial models are built. Often, frictions are simply shrugged off by arguing that for an individual “small” trader, their effects are nevertheless negligible and that frictionless models serve as useful approximations. As we shall see, sometimes this heuristic argument is indeed valid, but sometimes it can also lead us astray.

1.2 Approximation of price dynamics in markets with frictions

In real financial markets, the prices we observe are *endogeneously* determined outcomes of trades between investors. As mentioned above, frictions make continuous trading impracticable, so any investor can trade only finitely many times during her lifetime. Thus, since there are finitely many

investors, only finitely many changes in prices can occur during any finitely long period. This has the obvious corollary that prices cannot evolve continuously and, thus, all price changes must be *jumps*. However, typically stochastic processes with *continuous* paths, and in particular *diffusions*, are preferred as models of price dynamics, for the sake of tractability and parsimony—with the tacit understanding that they are approximations, in some sense. In the view of the discussion in the preceding section, we should be wary of such approximations, unless there is a compelling argument of validity.

An early approach to understand diffusion processes as approximations of price dynamics, starting from endogenous price formation in a *microscopic* market model, is due to Föllmer and Schweizer [10]. They develop a discrete-time model in which investors' aggregate demand and supply are matched in a sequence of temporary equilibria, determining equilibrium prices. They show that the equilibrium price process, when embedded into continuous time by interpolation, admits a diffusion process as its scaling limit. This setup with sequentially matched aggregate supply and demand can be seen as a model of a *batch market* (see [23, p. 255]). However, modern stock markets do not operate as batch markets, but instead as *continuous markets*, where trades are executed one by one at random dates. This leads to price dynamics that may be quite unlike those in batch markets. Thus, it would be desirable to understand if a result similar to [10] was valid in a reasonable model of a continuous market.

The field of *market microstructure* has studied extensively microscopic models of continuous markets, where the investors are indeed able to trade only in finite fashion leading to endogenous price dynamics that follow pure-jump processes. However, mathematically rigorous results on approximation of price dynamics in such models by continuous-path processes are scarce, perhaps in part due to analytical difficulties. A typical market microstructure model can be outlined as follows. There are n investors, indexed by $\mathbb{A} = \{1, \dots, n\}$, in the market, who trade the stock of some business. For any $a \in \mathbb{A}$, let $\varphi_{a,t}$ denote the number of shares held by investor $a \in \mathbb{A}$ at time $t \in [0, \infty)$. The main idea of the model is that each purchase and sale of shares prompts the price of the stock to change, depending on the number of shares traded. Specifically, there is a *price impact function* $r_n : \mathbb{R} \rightarrow \mathbb{R}$ that determines the change in the logarithmic price as a function of the traded amount. Thus, we have that price per share of the stock at time $t \in [0, T]$, denoted by P_t^n , is given by

$$\log P_t^n = \log P_0^n + \sum_{t \in (0, t]} \sum_{a \in \mathbb{A}} r_n(\varphi_{a,t} - \varphi_{a,t-}). \quad (1.4)$$

(The sums above involve only finitely many summands since the processes $\varphi_{a,\cdot}$, $a \in \mathbb{A}$ jump only finitely many times.) It is worth noting that the

definition of P^n through (1.4) is often non-trivial, since typically we want to allow for *feedback* from prices to the behavior of the investors, i.e. $\varphi_{a,t}$ depends on P_t^n , $t \in [0, t]$. Economically, we may think that the market is a so-called *dealership market* (a term coined by Garman [11]), where a *market maker* acts as a counterparty in each trade—the ordinary investors being unable to trade bilaterally—setting the price using r_n as a *pricing rule*.

Approximation of price dynamics by a process with continuous paths may be possible in this setup when the number of investors in the market is large and when individual investors' trades have very small impact on prices. To put it mathematically, we should study the behavior of P^n in the *large-market limit*, that is $n \rightarrow \infty$, under the assumption that, simultaneously, the price impact function r_n converges to zero. Since P^n is càdlàg process, we would aim to show convergence in law or in probability in the space $D([0, \infty), \mathbb{R})$ of càdlàg functions $[0, \infty) \rightarrow \mathbb{R}$, equipped with the locally uniform topology (see also p. 22 for a remark on the choice of the topology).

Previously, large-market limits for the price dynamics in microstructure models of the form outlined above have been established by Bayraktar, Horst, and Sircar [2], and Horst and Rothe [16]. The introduction of the former paper contains also a survey of some related results. The goal of these two papers appears to be to study approximation of price dynamics when the investors exhibit certain specific behavioral traits, namely *inertia* in [2], and usage of technical trading strategies that have *delayed dependencies* on past prices in [16]. In the models of both papers, each investor may buy or sell one share of the stock at a time, and the price impact function is assumed to be $r_n(x) = x/n$. Under some more specific model assumptions that capture the aforementioned behavioral traits (see [2, pp. 656–657] and [16, pp. 214–217], respectively), they show that $(P_t^n)_{t \in [0, \infty)}$ converges almost surely to the solution to a deterministic differential equation, denoted by $(p_t)_{t \in [0, \infty)}$. In [2], p solves an ordinary differential equation, whereas in [16], a so-called *delay* differential equation is needed to describe p (because of the price impact of technical trading strategies). Further, they show that the rescaled fluctuation process

$$\sqrt{n}(P_t^n - p_t), \quad t \in [0, \infty)$$

converges in law to a continuous stochastic process. In both papers, the limit is given by an integral equation driven by time-changed Brownian motions. Moreover, in [2], the limiting process can be approximated on long time scales by a *fractional* Ornstein–Uhlenbeck process—the long memory of which is due to investor inertia.

Paper [1] adopts a similar modeling approach as [2, 16], but focuses chiefly on understanding, when *diffusion processes* arise as large-market limits of price dynamics. The main assumptions are, roughly speaking, that $r_n(x) \sim cx/\sqrt{n}$, as $n \rightarrow \infty$ (see [1, p. 95] for a motivation for the choice

the of r_n) and, similarly to [2], that the investors' behavior depend on the past prices only through the most recent price, meaning that $\varphi_{a,t} - \varphi_{a,t-}$ depends on $(P_u^n)_{u \in [0,t]}$ through P_t^n . Effectively, the investors believe that the weak form of the *efficient markets hypothesis* holds true. Provided that investor strategies satisfy some further scaling assumptions in the aggregate level (see [I, pp. 95–97]), it is shown that P^n converges in law to a general diffusion process, the coefficients of which depend on some averaged behavioral characteristics of the investors (see [I, Theorem 2.1]). Note that, contrary to [2, 16], due to relatively larger price impact, P^n converges to a *stochastic* limit even without rescaling by \sqrt{n} .

In the model, in which the diffusion approximation is derived, investors interact with each other solely through *trading*. Paper [I] presents additionally a modified model in which investors have mean-field type of non-market interactions, which gives rise to *herd behavior*. Intermittently, this interaction triggers investors to “rush” to buy and sell the stock simultaneously, giving rise to rapid price movements. Due to this phenomenon, the price dynamics are quite different from the original model that lead to one-dimensional diffusion limit. Indeed, Proposition 3.1 of [I] establishes that price dynamics of the modified model can be approximated by a process with *stochastic volatility*. The volatility process is of the form $f(V_t)$, where V is an Ornstein–Uhlenbeck process and $f : \mathbb{R} \rightarrow (0, \infty)$ is a function that is connected to investors' tendency to herd. When this tendency is strong, f increases rapidly and causes the (limiting) price process to fluctuate wildly. In fact, in such cases it is possible to show that the logarithmic returns generated by the price process are *heavy-tailed*, in the sense that their second moments are infinite [I, Proposition 3.2] and/or that the tails decay polynomially (so-called *power-law* tails) [I, Proposition 3.3].

REMARK 1.5. It is, of course, a simplification that we consider a dealership market, as most modern stock markets use a *double auction* with electronic *limit order books* to determine market prices, instead of a purely market maker-based setting. However, models of such markets are inherently more complicated due to the need to describe the dynamics of the whole order book, which is a potentially infinite-dimensional object. At present, it still seems to be an open problem to formulate a model of a limit order book based on realistic assumptions (e.g. that investors' behavior is allowed to depend on the shape of the order book), in which emergence of diffusion can be established in a mathematically rigorous manner.

1.3 Arbitrage and hedging with small transaction costs

Transaction costs belong to the most common frictions that investors face when trading in real financial markets. Introduction of transaction costs inevitably precludes investors from using unreasonable trading strategies

with infinite trading volume, such as the delta hedge of a call option that was unraveled in Section 1.1.

The monograph of Kabanov and Safarian [19] gives a comprehensive survey of existing research on asset pricing with transaction costs. Here, we focus on the recent framework of Guasoni, Rásonyi, and Schachermayer [14], in which various problems related to arbitrage and hedging with transaction costs can be conveniently analyzed. To recall their model and main results, let $T \in (0, \infty)$ be a finite time horizon, and let us consider a market where a single stock is traded at exogeneous prices given by a continuous stochastic process $(S_t)_{t \in [0, T]}$ with values in $(0, \infty)$, such that $S_0 = s_0 \in (0, \infty)$. All trading is subject to proportional transactions costs. Specifically, for some small $\varepsilon > 0$, an investor incurs a transaction cost of εS_t units of cash per each traded share at time $t \in [0, T]$. We will discuss the following two basic questions:

- Are there arbitrage opportunities after transaction costs have been taken into account?
- Is it possible to superhedge non-path-dependent European derivatives in an efficient manner with transaction costs?

As we shall see, provided that S has a rather natural distributional property, known as *conditional full support*, the answer to both of these questions is negative.

First, we need to fix some concepts, starting with the formal definition of a trading strategy in the context of this model.

DEFINITION 1.6. We say that process $(\varphi_t)_{t \in [0, T]}$ is a *trading strategy*, if it is predictable, of finite variation, and satisfies $\varphi_0 = \varphi_T = 0$ a.s.

REMARK 1.7. We will consider solely strategies that are initiated and terminated without a position in the stock, amounting to the restriction $\varphi_0 = \varphi_T = 0$. When it comes to study of arbitrage, this proviso is reasonable. Unfortunately, for the purposes of hedging, it is more restrictive. However, we will confine ourselves to options that are settled in cash, so no issues will arise.

Suppose now that φ is a trading strategy. Then, since φ is of finite variation, we have $\varphi_t = \varphi_t^b - \varphi_t^s$ for any $t \in [0, T]$, where φ_t^b and φ_t^s denote the (cumulative) shares bought and sold, respectively, as of t . Thus, taking into account the incurred transaction costs, the terminal wealth generated by φ , starting from initial endowment $x \in \mathbb{R}$ is given by

$$V_S^\varepsilon(\varphi, x) := x + \int_0^T \varphi_t dS_t - \varepsilon \int_{[0, T]} S_t d(\varphi_t^b + \varphi_t^s),$$

where the stochastic integral with respect to S is defined pathwise, through integration by parts,

$$\int_0^T \varphi_t dS_t := - \int_{[0,T]} S_t d\varphi_t = \underbrace{\varphi_T S_T - \varphi_0 S_0}_{=0} - \int_{[0,T]} S_t d\varphi_t.$$

It is reasonable to think that the investor cannot allow the net value of her investments, i.e. the balance of her brokerage account to decline beyond some minimum level stipulated by her prime broker. In the event of such a decline, the prime broker would issue a *margin call*, asking her to liquidate her stock position or post more collateral into her *margin account*. The latter option would of course require an injection of external cash beyond the initial endowment, which we, by convention, do not allow. Hence, we exclude from our considerations trading strategies that may have arbitrarily small interim liquidation value.

DEFINITION 1.8. Trading strategy φ is *admissible* (with respect to S) if there exists $M \in (-\infty, 0)$ such that $V_S^\varepsilon(\varphi \mathbf{1}_{(0,t)}, 0) \geq M$ a.s. for all $t \in (0, T]$. We denote the class of such trading strategies by \mathcal{A}_S .

Let us move forward to the question, whether there are arbitrage opportunities in the model we have described. Following the principle that arbitrage is a strategy that has no downside risk and results in profits with positive probability, we fix the following definition.

DEFINITION 1.9. Let $\varepsilon \geq 0$. We say that S admits *arbitrage* with ε -sized proportional transaction costs, if there exists $\varphi \in \mathcal{A}_S$ such that $V_S^\varepsilon(\varphi, 0) \geq 0$ a.s. and $\mathbf{P}[V_S^\varepsilon(\varphi, 0) > 0] > 0$.

For practical purposes, this formal definition of arbitrage may be too rigid. If instead of eliminating the downside risk, it can be made very small, the strategy could be seen as an approximate arbitrage. This motivates the following more general notion.

DEFINITION 1.10. Let $\varepsilon \geq 0$. We say that S admits *free lunches with vanishing risk* with ε -sized proportional transaction costs, if there exists $\{\varphi^1, \varphi^2, \dots\} \subset \mathcal{A}_S$, such that

$$V_S^\varepsilon(\varphi^n, 0) \geq -\frac{1}{n} \quad \text{a.s. for all } n \in \mathbb{Z}_+,$$

a limit $\lim_{n \rightarrow \infty} V_S^\varepsilon(\varphi^n, 0)$ in $[0, \infty]$ exists a.s., and

$$\mathbf{P}\left[\lim_{n \rightarrow \infty} V_S^\varepsilon(\varphi^n, 0) > 0\right] > 0.$$

We will shortly see that arbitrage and free lunches with vanishing risk are ruled out, if the price process S can be coupled with a “shadow” price process that a.s. stays close to S , and transforms into a martingale by an equivalent change of the underlying probability measure.

DEFINITION 1.11. Let $\varepsilon > 0$. We say that $(\tilde{S}_t)_{t \in [0, T]}$ is an ε -consistent price system for S , if there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that \tilde{S} is an $(\mathbb{F}^S, \mathbf{Q})$ -martingale, and

$$|S_t - \tilde{S}_t| \leq \varepsilon S_t \quad \text{a.s. for all } t \in [0, T]. \quad (1.12)$$

REMARK 1.13. Since \mathbb{F}^S is augmented the usual way, we may actually assume that \tilde{S} is right-continuous. Hence, we may safely interchange the order of “a.s.” and “for all $t \in [0, T]$ ” in (1.12).

A key observation is that trading at prices given by a consistent price, without transaction costs, is always at least as profitable as at prices given by S , but with transaction costs.

LEMMA 1.14 (Dominance). *If S admits an ε -consistent price system \tilde{S} for some $\varepsilon > 0$, then for any trading strategy φ ,*

$$V_S^\varepsilon(\varphi, 0) \leq V_{\tilde{S}}^0(\varphi, 0) := \int_0^T \varphi_t d\tilde{S}_t \quad \text{a.s.}$$

PROOF. The assertion is essentially Lemma 2.1 of [13], although slightly differing notation and terminology are used therein. For the convenience of the reader, we reproduce the proof using our notation. First, let us write

$$V_S^\varepsilon(\varphi, 0) = V_{\tilde{S}}^0(\varphi, 0) + \int_0^T \varphi_t d(S_t - \tilde{S}_t) - \varepsilon \int_0^T S_t d(\varphi_t^b + \varphi_t^s).$$

Then, integrating by parts and using the definition of consistent price system (see also Remark 1.13), we have

$$\begin{aligned} \int_0^T \varphi_t d(S_t - \tilde{S}_t) &= \underbrace{\varphi_T(S_T - \tilde{S}_T) - \varphi_0(S_0 - \tilde{S}_0)}_{=0} - \int_0^T (S_t - \tilde{S}_t) d\varphi_t \\ &= \int_0^T (\tilde{S}_t - S_t) d\varphi_t^b + \int_0^T (S_t - \tilde{S}_t) d\varphi_t^s \\ &\leq \varepsilon \int_0^T S_t d(\varphi_t^b + \varphi_t^s), \end{aligned}$$

which implies the assertion. \square

By a suitable version of the fundamental theorem of asset pricing (e.g. Corollary 1.2 of [9]), any consistent price system admits no free lunches with vanishing risk when there are no transaction costs (i.e. $\varepsilon = 0$ in Definition 1.10). Thus, by Lemma 1.14, the following result is evident.

THEOREM 1.15 (No arbitrage). *Let $\varepsilon > 0$. If there exists an ε -consistent price system for S , then S does not admit free lunches with vanishing risk and, a fortiori, arbitrage opportunities with ε -sized proportional transaction costs.*

Now, we would of course want to know, for which price processes S can we find consistent price systems. Here enters the distributional property known as conditional full support that was mentioned in the beginning. Informally, this property dictates that at any given time, the conditional law of the future of the process, given the past, must have the largest possible support.

DEFINITION 1.16. The process S has *conditional full support* (CFS), if for any $\underline{t} \in [0, T)$ and for almost any $\omega \in \Omega$,

$$\text{supp}(\mathcal{L}[(S_t)_{t \in [\underline{t}, T]} | \mathcal{F}_{\underline{t}}^S](\omega)) = C_{S_{\underline{t}}(\omega)}([\underline{t}, T], (0, \infty)).$$

Here, $\mathcal{L}[(S_t)_{t \in [\underline{t}, T]} | \mathcal{F}_{\underline{t}}^S]$ is understood as a regular conditional law on $C([\underline{t}, T], (0, \infty))$. The definition of the support, ‘supp’, is given in Section 2.1. For example, geometric Brownian motion has CFS, which follows from Theorem 2.13 and Example 2.15, below. A thorough introduction to the basics of the CFS property is given in Section 2.2. The following result of Guasoni, Rásonyi, and Schachermayer [14, Theorem 1.2] now asserts that CFS implies existence of consistent price systems. Its proof [14, pp. 500–501] is based on a clever approximation with a discrete process, a so-called *random walk with retirement* (see Definition 2.3 of [14]).

THEOREM 1.17 (Consistent price systems). *If S has CFS, then there exists an ε -consistent price system for S for any $\varepsilon > 0$.*

REMARK 1.18. The CFS property is, by no means, necessary for the existence of consistent price systems. Evidently, any *bounded* continuous martingale in $(0, \infty)$ does not have CFS, but nevertheless has a trivial ε -consistent price system for any $\varepsilon > 0$, namely, the process itself.

Indeed, Bayraktar and Sayit [3] have introduced an alternative sufficient criterion for the existence of consistent price systems, which is weaker than CFS. However, when dealing with concrete price processes, it may often be easier to establish CFS rather than verify directly their criterion, the formulation of which, unlike CFS, involves stopping times and conditional probabilities with respect to stopped σ -algebras.

REMARK 1.19. By Theorems 1.15 and 1.17, the CFS property implies absence of arbitrage with transaction costs. To some extent, this implication holds even without transaction costs. Bender, Sottinen, and Valkeila [4, Theorem 6.12] show that the class of simple trading strategies that are based on stopping times that are *locally lower semicontinuous* (see [4, Definition 6.10]) functionals of the price process is devoid of arbitrage opportunities whenever the price process has CFS.

Next, we consider hedging of a European-type, non-path-dependent derivative $g(S_T)$, with cash settlement, where $g : (0, \infty) \rightarrow \mathbb{R}$ satisfies rather minimal conditions, made precise below. With ε -sized transaction

costs, the minimal initial endowment that enables one to superhedge this derivative is

$$p_\varepsilon(g(S_T)) := \inf\{x \in \mathbb{R} : \text{there is } \varphi \in \mathcal{A}_s \text{ such that } V_S^\varepsilon(\varphi, x) \geq g(S_T) \text{ a.s.}\}.$$

Our focus is on the behavior of this quantity when ε tends to zero, i.e. transaction costs are very small. To describe these asymptotics, we denote by $\hat{g} : (0, \infty) \rightarrow (-\infty, \infty]$ the *concave envelope* of g , which is the pointwise infimum of concave functions that dominate g (thus, a concave function itself). Using suitable consistent price systems, which can be constructed whenever S has CFS, Guasoni, Rásonyi, and Schachermayer [14, Theorem 1.3] show the following result, namely, that the cost of superhedging the derivative $g(S_T)$ with small transaction costs ($\varepsilon \downarrow 0$) is given by the concave envelope at s_0 (the initial price of the underlying stock).

THEOREM 1.20 (Superhedging). *Let $g : (0, \infty) \rightarrow \mathbb{R}$ be lower semicontinuous and bounded from below. If S has CFS, then*

$$p_\varepsilon(g(S_T)) \rightarrow \hat{g}(s_0) \quad \text{when } \varepsilon \downarrow 0.$$

Let us reflect on the reasonability of the cost of hedging, suggested by Theorem 1.20. To begin with, we may note that it depends on the price process only through the initial price s_0 , which is economically somewhat counterintuitive. Similarly to Section 1.1, let us consider again hedging of the call option $g(S_T) = (x - S_T)^+$. It is straightforward to check that then, $\hat{g}(x) = x$, $x \in (0, \infty)$. Thus, the cost of hedging tends to $\hat{g}(s_0) = s_0$, the initial price of the underlying stock, which means that with small transaction costs, it is not possible to improve on the trivial static superhedge of buying one share and selling it at maturity. For the writer of the option, this is clearly infeasible, as no reasonably rational investor would pay a premium of s_0 to buy a derivative that yields a payout inferior to the underlying stock. Similarly, one can check that the cost of hedging the *put option* $g(S_T) = (K - S_T)^+$ tends to K , which is again too high for practical purposes, as the payout of this option is always at most K .

This result refines the intuitive observation, made in Section 1.1, that the writer of the option cannot perfectly offset risk from the position by replicating it. Moreover, it tells us one would need to hike up the premium too much from the one suggested by the Black–Scholes approach to be able to finance a hedge that is applicable with frictions. In effect, the writer must carry a part of the risk arising from the written option herself, which is what happens in reality. Thus, we find that when it comes to hedging non-path-dependent European derivatives, the idea that the frictionless model somehow approximates the model with frictions is quite misleading.

To understand how universal the results of Theorems 1.15, 1.17, and 1.20 are, it is of interest to study which stochastic processes have the crucial CFS

property. The purpose of the included papers [II] and [III] is to widen the class of processes that are known to have CFS. To introduce the results of these papers, let $f : \mathbb{R} \rightarrow (0, \infty)$ be a bijection, e.g. $f(x) = e^x$. Paper [II] is devoted to the study of the CFS property of price processes of the form

$$S_t = f\left(H_t + \int_0^t k_s dX_s\right), \quad t \in [0, T],$$

where H and X are continuous processes and k is a measurable process. Theorem 3.1 of [II] asserts that S has CFS, if X is a standard Brownian motion, (H, k) are independent of X , and

$$\text{meas}(\{t \in [0, T] : k_t = 0\}) = 0 \quad \text{a.s.} \quad (1.21)$$

As a corollary, several common stochastic volatility models have CFS (see [II, Subsection 4.1]).

The assumption that (H, k) and X are mutually independent cannot be dropped in general (see Examples 3.1 and 3.2 of [II]). However, when (H, k) depend progressively on X (still a standard Brownian motion), S does have CFS under some strengthened assumptions. Namely, Theorem 3.2 of [II] implies that in the progressive case, we obtain CFS when $H_t = \int_0^t h_s ds$, $t \in [0, T]$, where h is bounded, and k is bounded from above and away from zero. This result can be used to establish CFS for price processes given by weak solutions to certain stochastic differential equations (see [II, Subsection 4.2]).

The result in the former case, with (H, k) and X mutually independent, generalizes beyond Brownian integrators. Namely, in this case, if X is a continuous process with CFS, k is of finite variation, and (1.21) holds, then S has CFS. This result is proved in [II, Theorem 3.3] under the stronger assumption that the paths of k never hit zero a.s., in lieu of (1.21). However, in Appendix A.2 it is shown that the condition (1.21) indeed suffices (see Theorem A.3).

Finally, paper [III] deals with price processes that are of the form

$$S_t = f\left(H_t + \int_{-\infty}^t g(t-s)k_s dW_s\right), \quad t \in [0, T],$$

where H is a continuous process, k is a càdlàg process, $g \in L^2((0, \infty))$, and W is standard Brownian motion, independent of (H, k) . If g and k satisfy some technical conditions that ensure that S has continuous modification (see [III, pp. 2–3]), k satisfies (1.21), and $\int_0^\varepsilon |g(s)|ds > 0$ for all $\varepsilon > 0$, then S has CFS [III, Theorem 3.1]. The main motivation of this result is that it establishes CFS for a subclass of so-called *Brownian semistationary processes*, introduced by Barndorff-Nielsen and Schmiegel [1]. Brownian semistationary processes form a potentially flexible class of models of price

dynamics that can incorporate simultaneously both heavy-tailed behavior and long memory effects. In many interesting cases they, however, fail to be semimartingales, which implies existence of free lunches with vanishing risk in the absence of frictions, as per [9, Theorem 7.2]. Nevertheless, by Theorems 1.15 and 1.17, small proportional transaction costs defeat these free lunches, whenever the CFS property is present.

2

Mathematical concepts and methods

2.1 Supports of the laws of stochastic processes

It is often useful to regard stochastic processes as random elements in suitable function spaces. This point of view leads us to consider laws of stochastic processes by means of probability measures on function spaces.

The concept of *support* describes where the probability mass of a probability measure is concentrated. In this work, we consider function spaces that are always (at least) separable metric spaces, so for our purposes it suffices to define the support in the context of such spaces.

DEFINITION 2.1. Suppose that E is a separable metric space and $\mu : \mathcal{B}(E) \rightarrow [0, 1]$ is a Borel probability measure on E . We call the smallest closed subset of E with μ -measure one the *support* of μ and denote it by $\text{supp}(\mu)$.

The unambiguity of Definition 2.1 is ensured by the following simple result.

LEMMA 2.2 (Existence and uniqueness). *Let E be a separable metric space and $\mu : \mathcal{B}(E) \rightarrow [0, 1]$ a Borel probability measure on E . Then, $\text{supp}(\mu)$ exists and is unique.*

PROOF. Let us denote by \mathcal{C}_μ the collection of closed sets $C \subset E$ such that $\mu(C) = 1$, and define $A := \bigcap \mathcal{C}_\mu$. Clearly, A is a closed subset of E , and there is no closed set $A' \subset E$ such that $A' \subsetneq A$ and $\mu(A') = 1$. Thus, it remains to show that $\mu(A) = 1$. By separability, E has a countable topological basis $\{U_i : i \in \mathbb{N}\}$. Hence, for any $C \in \mathcal{C}_\mu$,

$$E \setminus C = \bigcup_{i \in I_C} U_i,$$

where $I_C \subset \mathbb{N}$. Further, let us denote $I := \bigcup \{I_C : C \in \mathcal{C}_\mu\} \subset \mathbb{N}$. By associativity of unions,

$$E \setminus A = \bigcup_{C \in \mathcal{C}_\mu} \left(\bigcup_{i \in I_C} U_i \right) = \bigcup_{i \in I} U_i.$$

But for any $i \in I$, $U_i \subset E \setminus C$ for some $C \in \mathcal{C}_\mu$, so $\mu(U_i) = 0$. Hence, we have

$$1 \geq \mu(A) = 1 - \mu(E \setminus A) \geq 1 - \sum_{i \in I} \mu(U_i) = 1. \quad \square$$

Another way of characterizing $\text{supp}(\mu)$ is to say that it is the set of points around which balls of arbitrarily small radii have positive μ -measure. Below, we use the shorthand $B(f, \varepsilon)$ for the ball $\{g \in E : d_E(f, g) < \varepsilon\}$, where $d_E(\cdot, \cdot)$ is the metric of the underlying space E .

LEMMA 2.3 (Small-ball probabilities). *Let E be a separable metric space and $\mu : \mathcal{B}(E) \rightarrow [0, 1]$ a Borel probability measure on E . Then, $f \in \text{supp}(\mu)$ if and only if $\mu(B(f, \varepsilon)) > 0$ for all $\varepsilon > 0$.*

PROOF. Both necessity and sufficiency follow easily by contraposition. \square

Let us now turn to laws of stochastic processes. In what follows, we confine ourselves to continuous processes, although it would, at least in principle, be possible to study the supports of the laws of càdlàg processes.

Typically, it is not straightforward to characterize the support of the law of a continuous process. The support is intrinsically related to elusive *infinite-dimensional* properties of the law, which might be difficult to infer from the finite-dimensional marginal laws of the process. With Gaussian processes, however, supports can be characterized conveniently using the associated *reproducing kernel Hilbert spaces*.

Let $(X_t)_{t \in [u, v]}$ be a centered, continuous Gaussian process with covariance $K_X : [u, v] \times [u, v] \rightarrow \mathbb{R}$. Then, there exists (see e.g. [18, pp. 120–126]) a vector space $\mathcal{H}_X \subset C([u, v])$ and an inner product $\langle \cdot, \cdot \rangle_X : \mathcal{H}_X \times \mathcal{H}_X \rightarrow \mathbb{R}$ such that

- (i) $(\mathcal{H}_X, \langle \cdot, \cdot \rangle_X)$ is a Hilbert space,
- (ii) $K_X(\cdot, t) \in \mathcal{H}_X$ for any $t \in [u, v]$,
- (iii) $\langle f, K_X(\cdot, t) \rangle_X = f(t)$ for any $f \in \mathcal{H}_X$ and $t \in [u, v]$.

This inner product space is called the reproducing kernel Hilbert space (or alternatively, the *Cameron–Martin space*) of X . This name is motivated by the fact that \mathcal{H}_X is actually spanned by the covariance “kernel” K_X (or more precisely, by the functions $K_X(\cdot, t) \in \mathcal{H}_X$, $t \in [u, v]$). For the statement of the result that characterizes the support of $\mathcal{L}[X]$, the properties of the inner product are not really needed, they are here merely for the sake of completeness.

REMARK 2.4. Another characterization of the space \mathcal{H}_X follows from the Cameron–Martin theorem [18, Theorem 14.17]. Namely, \mathcal{H}_X is the set

of functions $f : [u, v] \longrightarrow \mathbb{R}$ such that

$$\mathcal{L}[(X_t + f(t))_{t \in [u, v]}] \sim \mathcal{L}[X]. \quad (2.5)$$

EXAMPLE 2.6 (Brownian motion). If X is standard Brownian motion on $[u, v]$, i.e. $K(t, s) = t \wedge s$ for any $t, s \in [u, v]$ and $X_u = 0$ a.s., then \mathcal{H}_X is the space of absolutely continuous functions $f : [u, v] \longrightarrow \mathbb{R}$ such that $f(u) = 0$ and $f' \in L^2([u, v])$. Further, the inner product $\langle \cdot, \cdot \rangle_X$ is given by

$$\langle f, g \rangle_X := \int_u^v f'(t)g'(t)dt, \quad f, g \in \mathcal{H}_X.$$

The following classical result of Kallianpur [21, Theorem 3] describes the support of $\text{Law}[X]$ in terms of \mathcal{H}_X . For the proof, see [21, pp. 118–119]. (Note that the proviso about the continuity of the covariance function therein is superfluous, see e.g. [18, Theorem 8.12].)

THEOREM 2.7 (Gaussian processes). *If X is a centered, continuous Gaussian process, then*

$$\text{supp}(\mathcal{L}[X]) = \overline{\mathcal{H}_X}, \quad (2.8)$$

where the bar denotes closure in the uniform topology of $C([u, v])$.

REMARK 2.9. The inclusion “ \supset ” in (2.8), which is actually the only part of the result that we shall need in the sequel, follows readily from the property (2.5). However, the proof of the reverse inclusion is more involved.

EXAMPLE 2.10 (Brownian motion, continued). When X is a standard Brownian motion on $[u, v]$, a simple induction argument shows that \mathcal{H}_X contains all polynomial functions without constant terms. Hence, by the Weierstrass approximation theorem, \mathcal{H}_X is dense in $C_0([u, v])$, and by Theorem 2.7 we have $\mathcal{L}[X] = C_0([u, v])$.

In addition to Gaussian processes, there is a plethora of studies on the supports of the laws of diffusion processes. A pioneering result in this line of research is the following support theorem of Stroock and Varadhan [24, Theorem 3.1].

THEOREM 2.11 (Diffusion processes). *Let $b, \sigma : [u, v] \times \mathbb{R} \longrightarrow \mathbb{R}$ be bounded Borel measurable functions, such that σ is bounded away from zero. If $(X_t)_{t \in [u, v]}$ is a weak solution to the stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_u = x \in \mathbb{R}.$$

and solution satisfies uniqueness in law, then

$$\text{supp}(\mathcal{L}[X]) = C_x([u, v], \mathbb{R}).$$

2.2 The conditional full support property

The notion of conditional full support was already touched upon in Section 1.3, in connection to a arbitrage and hedging with transaction costs. Here, we discuss basics of the conditional full support property from a more mathematical point of view.

Let $T \in (0, \infty)$, and suppose that we are given a continuous process $(X_t)_{t \in [0, T]}$ with values in some open interval $I \subset \mathbb{R}$ and a filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$. If X is adapted to \mathbb{F} , then for any $\underline{t} \in [0, T)$, the support of the regular conditional law of $(X_t)_{t \in [\underline{t}, T]}$, given $\mathcal{F}_{\underline{t}}$, is clearly at most $C_{X_{\underline{t}}}([\underline{t}, T], I)$. The case where the support actually *equals* $C_{X_{\underline{t}}}([\underline{t}, T], I)$ is of particular interest, aside from aforementioned applications, since it corresponds to the informal idea that at time \underline{t} any “future” is still possible. This motivates the notion of conditional full support, which we define here in a more general context than earlier in Definition 1.16.

DEFINITION 2.12. We say that the process X has *conditional full support* (CFS) with respect to the filtration \mathbb{F} , or briefly \mathbb{F} -CFS, in the state space I , if

- (i) X is adapted to \mathbb{F} ,
- (ii) for any $\underline{t} \in [0, T)$ and for almost any $\omega \in \Omega$,

$$\text{supp}(\mathcal{L}[(X_t)_{t \in [\underline{t}, T]} | \mathcal{F}_{\underline{t}}](\omega)) = C_{X_{\underline{t}}}([\underline{t}, T], I).$$

Below, we summarize some useful results concerning the CFS property.

THEOREM 2.13 (CFS basics). *Let X and \mathbb{F} be as above. Then, the following hold:*

- (i) *If $J \subset \mathbb{R}$ is another open interval and f is a homeomorphism between I and J , then X has \mathbb{F} -CFS in I if and only if $f(X)$ has \mathbb{F} -CFS in J .*
- (ii) *Let $\mathbb{G} := (\mathcal{G}_t)_{t \in [0, T]}$ be a filtration such that $\mathcal{G}_t \subset \mathcal{F}_t$ for all $t \in [0, T]$. If X has \mathbb{F} -CFS in I and it is adapted to \mathbb{G} , then it has also \mathbb{G} -CFS in I .*
- (iii) *The process X has \mathbb{F} -CFS in I if and only if it has CFS in I with respect to the usual augmentation of \mathbb{F} .*
- (iv) *The process X has \mathbb{F} -CFS in \mathbb{R} if and only if for any $\underline{t} \in [0, T)$, $f \in C_0([\underline{t}, T])$, $\varepsilon > 0$, and almost any $\omega \in \Omega$,*

$$\mathbf{P} \left[\sup_{t \in [\underline{t}, T]} |X_t - X_{\underline{t}} - f(t)| < \varepsilon \middle| \mathcal{F}_{\underline{t}} \right] (\omega) > 0. \quad (2.14)$$

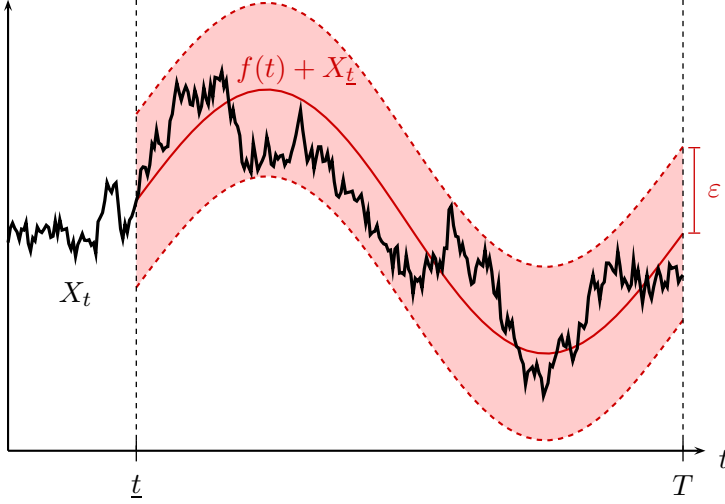


FIGURE 2.1. Condition (2.14) in Lemma 2.13 entails that the depicted event has positive \mathcal{F}_t -conditional probability.

- (v) Let $(Y_t)_{t \in [0, T]}$ be a continuous process, possibly on another probability space, such that $\mathcal{L}[Y] \sim \mathcal{L}[X]$. Then, X has \mathbb{F}^X -CFS in I if and only if Y has \mathbb{F}^Y -CFS in I .
- (vi) Let $(Y'_t)_{t \in [0, T]}$ be a continuous process, independent of X . If X has \mathbb{F}^X -CFS in \mathbb{R} , then $X + Y'$ has $\mathbb{F}^{X+Y'}$ -CFS in \mathbb{R} .

PROOF. Item (i) is proved in [II, Remark 2.1], (ii) is Corollary 2.1 of [II], (iii) is Lemma 2.3 of [II], and (iv) is Lemma 2.1 of [II]. Finally, (v) follows from Lemma 2.4 of [II] and Lemma 3.1 of [12], and (vi) is Lemma 3.2 of [12]. \square

EXAMPLE 2.15 (Brownian motion, continued). When X is a standard Brownian motion, we have by stationarity and independence of increments for any $\underline{t} \in [0, T)$, $f \in C_0([\underline{t}, T])$, and $\varepsilon > 0$,

$$\begin{aligned} \mathbf{P} \left[\sup_{t \in [\underline{t}, T]} |X_t - X_{\underline{t}} - f(t)| < \varepsilon \middle| \mathcal{F}_{\underline{t}}^X \right] &= \mathbf{P} \left[\sup_{t \in [\underline{t}, T]} |X_t - X_{\underline{t}} - f(t)| < \varepsilon \right] \\ &= \mathbf{P} \left[\sup_{t \in [0, T - \underline{t}]} |X_t - f(t)| < \varepsilon \right] > 0 \end{aligned}$$

almost surely, where the final inequality follows from Example 2.10 and Lemma 2.3. Hence, X has \mathbb{F}^X CFS in \mathbb{R} .

EXAMPLE 2.16 (Approximation). Any continuous process with values in \mathbb{R} can be “approximated” by a continuous process with CFS. To see this,

let $(Z_t)_{t \in [0, T]}$ be an arbitrary continuous process and define for any $\varepsilon > 0$, process $Z^\varepsilon = Z + \varepsilon W$, where W is standard Brownian motion independent of Z . Then, by Example 2.15 and Lemma 2.13(vi), Z^ε has $\mathbb{F}^{Z^\varepsilon}$ -CFS in \mathbb{R} , and moreover, $Z^\varepsilon \rightarrow Z$ a.s. when $\varepsilon \rightarrow 0$.

REMARK 2.17. A brief survey of processes that are known to have CFS is given in [II, p. 652].

It is apt to conclude the discussion of the CFS property by stating the following remarkable result due to Guasoni, Rásonyi, and Schachermayer [14, Lemma 2.9]. It asserts that, whenever the CFS property is in force, the condition (ii) in Definition 2.12 actually holds not only for all deterministic times, but also for all *stopping times* with values in $[0, T]$. This result is crucial in proving that CFS implies existence of consistent price systems, see [14, p. 516].

THEOREM 2.18 (Strong CFS). *If X has \mathbb{F}^X -CFS in I , then for any stopping time τ , with values in $[0, T]$, and for almost any $\omega \in \{\tau < T\}$,*

$$\text{supp}(\mathcal{L}[(X_t)_{t \in [\tau, T]} | \mathcal{F}_\tau](\omega)) = C_{X_\tau(\omega)}([\tau(\omega), T], I).$$

2.3 Pure-jump Markov processes and their approximation

The formulation and analysis of the microscopic market model, introduced in paper [I], relies on *pure-jump Markov processes*, the basic properties of which we introduce in this section.

First, recall that stochastic process $(X_t)_{t \in [0, \infty)}$, with values in \mathbb{R}^d , is said to be *Markov*, if for any $\underline{t} \in [0, \infty)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and almost any $\omega \in \Omega$,

$$\mathbf{P}[X_t \in B | \mathcal{F}_{\underline{t}}^X](\omega) = \mathbf{P}[X_t \in B | X_{\underline{t}}](\omega).$$

If additionally X is càdlàg, and its paths are constant apart from isolated jumps, we say that it is a *pure-jump Markov process*.

For any $x \in \mathbb{R}^d$, let us denote by \mathbf{P}_x the (regular) conditional law $\mathbf{P}[\cdot | X_0 = x]$ and by \mathbf{E}_x the expectation with respect to \mathbf{P}_x . (We may assume that X is defined on the canonical space $D([0, \infty), \mathbb{R}^d)$, so no problems with the existence of \mathbf{P}_x will arise.) Let us define τ to be the time of the first jump of X , i.e. $\tau := \inf\{t > 0 : X_t \neq X_{t-}\}$. The function $\lambda : \mathbb{R}^d \rightarrow (0, \infty)$, defined by

$$\lambda(x) := \frac{1}{\mathbf{E}_x[\tau]}, \quad x \in \mathbb{R}^d,$$

is called the *intensity function* of X . Further, we define the *jump transition kernel* ν of X by

$$\nu(x, B) := \mathbf{P}_x[X_\tau - x \in B], \quad x \in \mathbb{R}^d, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Finally, the product $\lambda\nu$ is called the *rate kernel* of X . The rate kernel in fact determines the conditional laws \mathbf{P}_x , $x \in \mathbb{R}^d$ unambiguously (see [20, Theorem 12.17]).

Next, we present a “recipe” for a general pure-jump Markov process. To this end, let $f : \mathbb{R}^d \times [0, 1] \rightarrow \mathbb{R}^d$ be a measurable function such that for any $x \in \mathbb{R}^d$ and for almost any $y \in [0, 1]$,

$$x \neq f(x, y), \quad (2.19)$$

and let $\lambda : \mathbb{R}^d \rightarrow (0, \infty)$ be a bounded measurable function. Further, let ξ_1, ξ_2, \dots and $\gamma_1, \gamma_2, \dots$ be mutually independent sequences of i.i.d. random variables, where $\xi_1 \sim U(0, 1)$ and $\gamma_1 \sim \text{Exp}(1)$. Additionally, let Y_0 be a random initial value in \mathbb{R}^d , independent of the aforementioned sequences of random variables. Let us now define $\tau_0 := 0$,

$$\tau_k := \tau_{k-1} + \frac{\gamma_k}{\lambda(Y_{k-1})}, \quad Y_k := f(Y_{k-1}, \xi_k), \quad k \in \mathbb{Z}_+, \quad (2.20)$$

and finally,

$$X_t := \sum_{k \in \mathbb{N}} Y_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t), \quad t \in [0, \infty). \quad (2.21)$$

Using some standard results, to be found e.g. in Chapters 8 and 12 of [20], we may show that X is Markov and characterize its rate kernel.

THEOREM 2.22 (Construction). *Let $(X_t)_{t \in [0, \infty)}$ be defined by (2.20) and (2.21). Then, X is a pure-jump Markov process with rate kernel $\lambda\nu$, where*

$$\nu(x, B) := \text{meas}(\{y \in [0, 1] : f(x, y) - x \in B\})$$

for any $x \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

PROOF. By Proposition 8.6 of [20], it is clear that Y_0, Y_1, Y_2, \dots is a Markov chain with transition kernel ν . Next, note that

$$\lim_{k \rightarrow \infty} \tau_k \geq \sum_{k \in \mathbb{Z}_+} \frac{\gamma_k}{\|\lambda\|_\infty} = \infty \quad \text{a.s.}$$

e.g. by Proposition 4.14 of [20], and that for any $x \in \mathbb{R}^d$, we have $\nu(x, \{0\}) = 0$ by (2.19). Thus, the assertion follows from Theorem 12.18 of [20]. \square

EXAMPLE 2.23 (Compound Poisson process). A *compound Poisson process* with jump rate $\lambda_0 > 0$ and jump distribution F_J on \mathbb{R} , where $F_J(0) - F_J(0-) = 0$, can be constructed using the “recipe” above as follows. Set $\lambda(x) := \lambda_0$ for all $x \in \mathbb{R}$ and

$$f(x, y) := x + F_J^{-1}(y), \quad (x, y) \in \mathbb{R} \times [0, 1],$$

where $F_J^{-1}(y) := \inf\{x \in \mathbb{R} : F_J(x) \geq y\}$. Then, we may notice that the jump transition kernel satisfies $\nu(x, dy) = F_J(dy)$ for any $x \in \mathbb{R}$, as we would expect.

We now turn to the problem of approximating pure-jump Markov processes in law by diffusion processes—or in other words, determining when a sequence of pure-jump Markov processes converges in law to a diffusion. For this purpose, we regard both pure-jump Markov processes and diffusions as random elements of the space $D([0, \infty), \mathbb{R}^d)$. Usually, $D([0, \infty), \mathbb{R}^d)$ is equipped with the *Skorohod topology* (see e.g. [17, pp. 325–346] for a detailed description) that turns it into a separable topological space with complete metrization (i.e. a *Polish space*). However, since we are studying convergence to a continuous process, we may as well think of $D([0, \infty), \mathbb{R}^d)$ being equipped with the stronger locally uniform topology, induced by uniform convergence on compact sets (see e.g. [17, Proposition IV.1.17]).

A criterion for the convergence of pure-jump Markov processes to a diffusion can be conveniently formulated in terms of their rate kernels. Before stating the result, we need to introduce an assumption that ensures that the diffusion process in the limit exists and that its law is unambiguous. To this end, let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow M_d(\mathbb{R})$ be continuous functions (where the space $M_d(\mathbb{R})$ is equipped with the norm topology).

ASSUMPTION 2.24 (Existence and uniqueness of solutions). For any $x \in \mathbb{R}^d$, there exists a unique (in law) weak solution $(X_t^x)_{t \in [0, \infty)}$ to stochastic differential equation

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0^x = x,$$

where $(W_t)_{t \in [0, \infty)}$ is a standard Brownian motion in \mathbb{R}^d .

The following rendition of Theorem IX.4.21 of [17] gives now sufficient conditions for the convergence.

THEOREM 2.25 (Diffusion approximation). *Let X^1, X^2, \dots be pure-jump Markov processes in \mathbb{R}^d with bounded rate kernels μ_1, μ_2, \dots such that for any $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}_+$, we have $\int_{\mathbb{R}^d} \mu_n(x, dy) \|y\|_{\mathbb{R}^d}^2 < \infty$. Moreover, suppose that following convergence conditions hold:*

- (i) *There exists continuous functions $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow M_d(\mathbb{R})$, satisfying Assumption 2.24, such that*

$$\int_{\mathbb{R}^d} \mu_n(\cdot, dy) y \xrightarrow[n \rightarrow \infty]{\text{u.o.c.}} b(\cdot), \quad \int_{\mathbb{R}^d} \mu_n(\cdot, dy) y y^T \xrightarrow[n \rightarrow \infty]{\text{u.o.c.}} \sigma(\cdot) \sigma(\cdot)^T.$$

- (ii) *For any $\varepsilon > 0$,*

$$\int_{\{\|x\|_{\mathbb{R}^d} > \varepsilon\}} \mu_n(\cdot, dy) \|y\|_{\mathbb{R}^d}^2 \xrightarrow[n \rightarrow \infty]{\text{u.o.c.}} 0.$$

- (iii) *There exists a random variable ξ such that $X_0^n \xrightarrow[n \rightarrow \infty]{\text{law}} \xi$.*

Then,

$$X^n \xrightarrow[n \rightarrow \infty]{\text{law}} X \quad \text{in } D([0, \infty), \mathbb{R}^d),$$

where $(X_t)_{t \in [0, \infty)}$ is the unique (in law) weak solution to the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = \xi,$$

where $(W_t)_{t \in [0, \infty)}$ is standard Brownian motion in \mathbb{R}^d .

2.4 Heavy tails and regular variation

In Section 1.2, it was mentioned that the large-market limit of the price dynamics of the modified model in [1] may exhibit *heavy-tailed* log returns. In this section, we recall briefly some relevant facts concerning heavy tails.

Let X be a random variable with values in \mathbb{R} . Informally, we say that X , or more precisely its distribution, has *heavy tails* if tail probability $\mathbf{P}[|X| > x]$ decays “slowly” when $x \rightarrow \infty$. While there is no unanimity in the literature over what qualifies as “slow” decay, random variables with at most *polynomially* decaying tails—as opposed to e.g. *exponentially* decaying tails, like Gaussian—are virtually always considered heavy-tailed.

The rate of decay of the tail probabilities is closely connected to finiteness of moments, since by Fubini’s theorem for any $p > 0$,

$$\mathbf{E}[|X|^p] = p \int_0^\infty x^{p-1} \mathbf{P}[|X| > x] dx.$$

Thus, polynomial decay rates are critical. Let $c > 0$, and $x_0 > 0$. If

$$\mathbf{P}[|X| > x] \geq cx^{-\alpha}, \quad x \geq x_0 \tag{2.26}$$

for some $\alpha \in (0, p]$, then $\mathbf{E}[|X|^p] = \infty$, while if

$$\mathbf{P}[|X| > x] \leq cx^{-\alpha}, \quad x \geq x_0 \tag{2.27}$$

for some $\alpha \in (p, \infty)$, then $\mathbf{E}[|X|^p] < \infty$.

A wide, and extensively-studied class of random variables with polynomially decaying tails—also dubbed *power-law* tails—can be defined through the notion of *regular variation*.

DEFINITION 2.28. We say that the random variable X is *regularly varying* with index $\alpha > 0$, denoted by $X \in \text{RV}(\alpha)$, if

$$\mathbf{P}[|X| > x] \sim L(x)x^{-\alpha}, \quad x \rightarrow \infty, \tag{2.29}$$

where $L : (0, \infty) \rightarrow (0, \infty)$ is a *slowly varying* function, i.e. for any $c > 0$,

$$\lim_{x \rightarrow \infty} \frac{L(cx)}{L(x)} = 1.$$

EXAMPLE 2.30. Function $x \mapsto c \cdot (\log x)^\kappa$ is slowly varying for any $c > 0$ and $\kappa \in \mathbb{R}$, whereas functions $x \mapsto \sin x$ and $x \mapsto x^\kappa$, where $\kappa \in \mathbb{R} \setminus \{0\}$, are not slowly varying.

The role of the slowly varying function L in (2.29) might seem a bit opaque at first, so let us look into its properties. Somewhat surprisingly, all slowly varying functions have a specific, semi-explicit form, given by the following result, known as *Karamata's representation theorem* (for the proof, see e.g. [5, p. 13]).

THEOREM 2.31 (Representation). *If $L : (0, \infty) \rightarrow (0, \infty)$ is a slowly varying function, then*

$$L(x) = c(x) \exp \left(\int_0^x \frac{h(y)}{y} dy \right), \quad x > 0,$$

for some measurable functions c and h such that $\lim_{x \rightarrow \infty} c(x) > 0$ and $\lim_{x \rightarrow \infty} h(x) = 0$.

Since the function h can be bounded by an arbitrary small constant ultimately, Karamata's representation theorem implies that for any $\varepsilon > 0$, there exists $x_0 > 0$ such that

$$x^{-\varepsilon} \leq L(x) \leq x^\varepsilon, \quad x \geq x_0.$$

Thus, in regard to polynomial decay, the slowly varying part in (2.29) is merely a perturbation. Consequently, if $X \in \text{RV}(\alpha)$ for some $\alpha > 0$, then for any $\varepsilon > 0$ and $\delta > 0$, there exists $x_0 > 0$ such that

$$(1 - \delta)x^{-(\alpha+\varepsilon)} \leq \mathbf{P}[|X| > x] \leq (1 + \delta)x^{-(\alpha-\varepsilon)}, \quad x \geq x_0.$$

Hence, by criteria (2.26) and (2.27), we find that $\mathbf{E}[|X|^p] = \infty$ for all $p \in (\alpha, \infty)$ and $\mathbf{E}[|X|^p] < \infty$ for all $p \in (0, \alpha)$.

Finally, in the proof of Proposition 3.3 of [I], it is argued that the product of a Gaussian random variable and a regularly varying random variable, provided they are mutual independent, is regularly varying. This argument is valid by the following result, known as *Breiman's lemma*. Originally, Breiman [7, pp. 326–327] proved this result in the case $\alpha \in (0, 1)$, while a complete proof covering all $\alpha > 0$ is given e.g. in [8, Theorem 3.5(v)].

LEMMA 2.32 (Products). *Suppose that $X \in \text{RV}(\alpha)$ for some $\alpha > 0$, $Y \in L^{\alpha+\varepsilon}(\mathbf{P})$ for some $\varepsilon > 0$, and that X is independent of Y . Then, $XY \in \text{RV}(\alpha)$. More precisely,*

$$\mathbf{P}[|XY| > x] \sim \mathbf{E}[|Y|^\alpha] \mathbf{P}[|X| > x], \quad x \rightarrow \infty.$$

A

Appendix

A.1 Proof of Lemma 2.1 of [I]

To save space, the proof of Lemma 2.1 of [I] was omitted in the original paper. For the convenience of the reader, we present here the omitted details.

PROOF OF LEMMA 2.1 OF [I]. We begin by considering certain functions and their measurability. Firstly, for each $x \in \mathbb{R}$, let $\{0 = i_{x,0} \leq i_{x,1} \leq \dots \leq i_{x,n} = 1\}$ be such that $i_{x,a} - i_{x,a-1} = \lambda_a(x)/\lambda_{\mathbb{A}_n}(x)$. Since the functions λ_a , $a \in \mathbb{A}_n$ are continuous, one can check that

$$g(x, y) := \sum_{a=1}^n a \mathbf{1}_{[i_{x,a-1}, i_{x,a})}(y) \quad (\text{A.1})$$

defines a Borel measurable function $g : \mathbb{R} \times [0, 1] \longrightarrow \mathbb{A}_n$. Secondly, because the function $(x, s) \mapsto e_a^n(x, s)$ is Borel measurable for every fixed $a \in \mathbb{A}_n$ and the set \mathbb{A}_n is finite, it follows that $(a, x, s) \mapsto e_a^n(x, s)$ and consequently $(a, x, s) \mapsto r_n(e_a^n(x, s), x)$ are measurable with respect to $\mathcal{P}(\mathbb{A}_n) \otimes \mathcal{B}(\mathbb{R}^2)$, where $\mathcal{P}(\mathbb{A}_n)$ stands for the power set of \mathbb{A}_n .

Now, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space that carries mutually independent i.i.d. sequences $(\eta_k)_{k=0}^\infty$ and $(\gamma_k)_{k=1}^\infty$, such that $\eta_0 \sim \text{U}(0, 1)$ and $\gamma_1 \sim \text{Exp}(1)$. Recall, that by Lemma 3.21 of [20], there exist Borel measurable functions $\hat{u}, \tilde{u} : [0, 1] \rightarrow [0, 1]$, such that $\hat{\eta}_k := \hat{u}(\eta_k)$ and $\tilde{\eta}_k := \tilde{u}(\eta_k)$ are independent $\text{U}(0, 1)$ random variables. Let us define $P_0 := F_{P_0}^{-1}(\eta_0)$ and $\xi_k := F_\xi^{-1}(\hat{\eta}_k)$ for all $k \in \mathbb{Z}_+$, where $F^{-1}(y) := \inf\{x : F(x) \geq y\}$. Additionally, define recursively

$$A_k := g(P_{k-1}, \tilde{\eta}_k), \quad P_k := r_n(e_{A_k}^n(P_{k-1}, \xi_k), P_{k-1}), \quad k \in \mathbb{Z}_+. \quad (\text{A.2})$$

With these specifications, recalling [I, eq. 4] and [I, Assumption 2.2], the assertion follows now from Theorem 2.22. \square

A.2 Extension of Theorem 3.3 of [II]

As mentioned in Section 1.3, the assumption in Theorem 3.3 of [II] concerning the integrand k can be weakened as follows.

THEOREM A.3 (Conditional full support). *Let $(H_t)_{t \in [0, T]}$ be a continuous process, $(k_t)_{t \in [0, T]}$ a process of finite variation, and $(X_t)_{t \in [0, T]}$ a continuous process that has CFS with respect to the filtration*

$$\mathcal{G}_t := \sigma\{X_s : s \in [0, t]\} \vee \sigma\{H_u, k_u : u \in [0, T]\}, \quad t \in [0, T]$$

in \mathbb{R} . Further, define

$$Z_t := H_t + \int_0^t k_s dX_s, \quad t \in [0, T].$$

If $\text{meas}(\{t \in [0, T] : k_t = 0\}) = 0$ a.s., then Z has $(\mathcal{G}_t)_{t \in [0, T]}$ -CFS in \mathbb{R} .

REMARK A.4. Obviously, X has $(\mathcal{G}_t)_{t \in [0, T]}$ -CFS, when it has \mathbb{F}^X -CFS and it is independent of (H, k) .

PROOF. Fix $\underline{t} \in [0, T)$. By Lemma 4.3 of [III], for a.a. $\omega \in \Omega$, the range of the integral operator $K_\omega : L^2([\underline{t}, T]) \rightarrow C_0([\underline{t}, T])$, defined by

$$(K_\omega f)(t) := H_{\underline{t}}(\omega) - H_t(\omega) + \int_{\underline{t}}^t k_s(\omega) f(s) ds, \quad t \in [\underline{t}, T],$$

is dense in $C_0([\underline{t}, T])$. Thus, it suffices to show that

$$\mathbf{P} \left[\sup_{t \in [\underline{t}, T]} |Z_t - Z_{\underline{t}} - (K.f)(t)| < \varepsilon \middle| \mathcal{G}_{\underline{t}} \right] > 0 \quad \text{a.s.}$$

for any $\varepsilon > 0$ and $f \in L^2([\underline{t}, T])$. But integrating by parts we may estimate for any $t \in [0, T]$,

$$|Z_t - Z_{\underline{t}} - (K.f)(t)| \leq \left(\sup_{t \in [\underline{t}, T]} |k_t| + \text{TV}_{[\underline{t}, T]}(k) \right) \cdot \sup_{t \in [\underline{t}, T]} |X_t - X_{\underline{t}} - F(t)|,$$

where $\text{TV}_{[\underline{t}, T]}(k)$ stands for the total variation of the path of k on the interval $[\underline{t}, T]$ and $F(t) := \int_{\underline{t}}^t f(s) ds$. To complete the proof, we could now proceed similarly as in the proof of Theorem 3.3 of [II]. \square

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